On the Stability of Anti-de Sitter Spacetime

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We present a detailed analysis for the classical stability of a four dimensional Anti-de Sitter spacetime (AdS$_4$) by decomposing the first-order perturbations of a spherical symmetric gravitational field into the so called tensor harmonics which transform as irreducible representative of the rotation group (Regge-Wheeler decomposition). It is shown that there is no nontrivial stationary perturbation for the angular momentum $l < 2$. The stability analysis forces the frequency of the gravitational modes to be constrained in a way that the frequency of scalar modes are constrained.

I. INTRODUCTION

A *priori* there is no trivial reason to believe that a non-hyperbolic spacetime like the Anti-de Sitter (AdS) is a stable configuration. Stability of spacetimes with negative cosmological constant against the quantum scalar field [1, 2] and in the context of supergravity [3, 4] has already been studied in great details. The reincarnation of the AdS spacetime in the form of the AdS/CFT correspondence [5, 6, 7] gives a good excuse to study the stability of the AdS spacetime in much more details. In the present work we study thoroughly the classical stability of the AdS$_4$ against gravitational perturbation modulo to the ambiguity of defining a gauge invariant boundary conditions at the timelike spatial infinity of the AdS. To this aim we use the Regge-Wheeler metric decomposition and gauge [10] which used initially to determine the stability of the Schwarzschild-type black holes. In this approach the perturbation on a spherically background is decomposed into its normal vibrational modes using tensor spherical harmonics. The perturbations superposed on the AdS$_4$ background metric are the same as those given by Regge and Wheeler [10], consisting of odd and even parity categories, and with the exponential time dependence, i.e., $\exp\left(-\mathrm{i}\omega t\right)$. Thus perturbation with positive imaginary frequencies are responsible for instability. We show that the frequency of the modes are discrete and real positive. There is no non-radiative mode and therefore the perturbed pure AdS spacetime is full of gravitational radiation.

The organization of our paper is as follows: In Sec. II we review perturbation around curved background. In Sec. III geometrical and group theoretical features of the four dimensional Anti-de sitter (AdS$_4$) are studied. We parameterize AdS in the static spherical symmetric coordinates which cover the whole spacetime. We then perturbed this metric in accordance to Regge-Wheeler decomposition of the metric. In Sec. IV the tensor spherical harmonics are studied in great details. General form of the perturbation metric in Regge-Wheeler decomposition is derived for both the odd (magnetic) and the even (electric) modes. After gauge fixing we derive the perturbed Einstein equations and from there the governing equations of motion for the monopole ($l = 0$), the electric/magnetic dipole ($l = 1$) and the radiative ($l \geq 2$) modes are studies in details. We will argue why no non-radiative mode exists in the pure AdS. The Schrödinger-type equation of motion is derived for both magnetic and electric radiative modes. The magnetic and electric effective potentials in the absence of any gravitational scatterer (mass) are identical which is mainly due to the symmetry of the AdS$_4$ group and its invariant Casimir operator. Section V is devoted to discussion of stability of radiative modes. We find the exact solution for the radiative modes. In order to have a well-behaved solution in entire range of radial coordinate, the frequency of the modes appears in discrete tones very much similar to the frequency of the massless minimally coupled quantum scalar field in the AdS$_4$ [2]. Finally in Sec. VI we discuss our results and future directions.

II. SMALL FLUCTUATIONS AROUND THE CURVED BACKGROUND

Let us consider a general D-dimensional action with a cosmological constant $Ł_D$

$$\mathcal{L}_D = \sqrt{^{(D)}g}\left[^{(D)}R + 2\Lambda_D\right] , \tag{1}$$

The field equations then take the following simple form

$$^{(D)}R_{MN} = -\mathrm{L}_D g_{MN}\,. \tag{2}$$

A general perturbation around this background may be written as

$$ds^2 = \hat{g}_{MN}dx^M dx^N = (g_{MN} + h_{MN})dx^M dx^N\,, \tag{3}$$

where h_{MN} is a small perturbation from the background g_{MN}. Let us denote "hat" quantities to be those associated with $\hat{g}_{MN}$ metric and quantities without "hat" to those associated with g_{MN}. Christoffel symbols and Ricci tensor are defined as follows

$$
\begin{aligned}
\hat{\Gamma}^M_{NP} &= \frac{1}{2}\hat{g}^{MQ}(\partial_N \hat{g}_{QP} + \partial_P \hat{g}_{NQ} - \partial_P \hat{g}_{NP})\,, \\
\hat{R}_{MN} &= \partial_P \hat{\Gamma}^P_{MN} - \partial_M \hat{\Gamma}^P_{NP} + \hat{\Gamma}^P_{MN}\hat{\Gamma}^Q_{PQ} - \hat{\Gamma}^P_{QN}\hat{\Gamma}^Q_{PM}\,,
\end{aligned}
\tag{4}
$$

where

$$\hat{g} = \det(\hat{g}_{MN}) = \det(g_{MN}) + O(h^{\geq 2}) = g\,. \tag{5}$$

In terms of g_{MN} and up to the first order in h_{MN} one has

$$\hat{\Gamma}^M_{NP} = \Gamma^M_{NP} - \frac{1}{2}h^{MQ}(\partial_N g_{QP} + \partial_P g_{NQ} - \partial_Q g_{NP}) + \frac{1}{2}g^{MQ}(\partial_N h_{QP} + \partial_P h_{NQ} - \partial_Q h_{NP}) \tag{6}$$

Using the fact that

$$\nabla_\alpha f_{\beta\gamma} \equiv \partial_\alpha f_{\beta\gamma} - \Gamma^\delta_{\alpha\beta}f_{\delta\gamma} - \Gamma^\delta_{\alpha\gamma}f_{\delta\beta}\,, \tag{7}$$

and

$$\nabla_\alpha f^{\beta\gamma} \equiv \partial_\alpha f_{\beta\gamma} + \Gamma^\beta_{\alpha\delta}f^{\delta\gamma} + \Gamma^\gamma_{\alpha\delta}f_{\delta\beta}\,, \tag{8}$$

with

$$\Gamma^\alpha_{\beta\gamma} \equiv \frac{1}{2}g^{\alpha\delta}(\partial_\beta g_{\delta\gamma} + \partial_\gamma g_{\beta\delta} - \partial_\delta g\beta\gamma)\,, \tag{9}$$

One can rewrite (6) in a compact form

$$\hat{\Gamma}^M_{NP} = \Gamma^M_{NP} + \frac{1}{2}g^{MQ}(\nabla_N h_{QP} + \nabla_P h_{NQ} - \nabla_Q h_{NP}) = \Gamma^M_{NP} + \delta\Gamma^M_{NP}\,. \tag{10}$$

Using the above relation and keeping only the first order in h_{MN} one finds

$$\hat{R}_{MN} = R_{MN} + \nabla_P(\delta\Gamma^P_{MN}) - \nabla_N(\delta\Gamma^P_{PM})\,, \tag{11}$$

or

$$\hat{R}_{MN} = R_{MN} + \frac{1}{2}g^{PQ}(\nabla_P \nabla_M h_{QN} + \nabla_P \nabla_N h_{MQ} - \nabla_M \nabla_N h_{PQ} - \nabla_P \nabla_Q h_{MN})\,. \tag{12}$$

Thus

$$\delta R_{MN} = \frac{1}{2}\left[\square_T h_{MN} - \nabla^A \nabla_M h_{AN} - \nabla^A \nabla_N h_{MA} + \nabla_M \nabla_N h^A{}_A\right] = -\mathrm{L}_D h_{MN}\,, \tag{13}$$

where $\square_T = g^{MN}\nabla_M \nabla_N$ is the covariant form of the d'Alembert's operator, in which

$$
\begin{aligned}
\nabla_A \nabla_B h_{MN} =\ & \left(\partial_A \partial_B - \Gamma^C_{AB}\partial_C\right)h_{MN} + \left(\Gamma^C_{AM}\Gamma^E_{BN} + \Gamma^C_{AN}\Gamma^E_{BM}\right)h_{CE} \\
& - \left(\Gamma^C_{BN}\partial_A + \Gamma^C_{AN}\partial_B + \partial_A \Gamma^C_{BN} - \Gamma^E_{AB}\Gamma^C_{EN} - \Gamma^E_{AN}\Gamma^C_{EB}\right)h_{MC} \\
& - \left(\Gamma^C_{BM}\partial_A + \Gamma^C_{AM}\partial_B + \partial_A \Gamma^C_{BM} - \Gamma^E_{AB}\Gamma^C_{EM} - \Gamma^E_{AM}\Gamma^C_{EB}\right)h_{CN}\,.
\end{aligned}
\tag{14}
$$

Now having (13) in our disposal we can calculate the perturbation modes around an AdS$_4$. Before getting to this, however, we first review AdS$_4$ and its geometrical and group theoretical properties.

III.   FOUR DIMENSIONAL ANTI-dE SITTER SPACETIME (AdS$_4$)

AdS$_4$ can be visualized as the four dimensional hyperboloid

$$- U^2 - V^2 + X^2 + y^2 + Z^2 = -\ell^2 \,, \tag{15}$$

embedded in five dimensional flat Minkowski spacetime

$$ds^2 = -dU^2 - dV^2 + dX^2 + dY^2 + dZ^2 = \eta_{ab} X^a X^b \,, \tag{16}$$

where ℓ is the radius of the AdS$_4$ related to the Ricci scalar curvature by $\ell^{-2} = \left({}^{(4)}R/12 \right) = -(\text{Ł}_4/3)$ and $\eta_{ab} = \mathrm{diag}(-,-,+,+,+)$.

To parameterize this chart we process by choosing the following coordinates

$$\begin{aligned}
X &= r \sin\theta \cos\varphi \,, \\
Y &= r \sin\theta \sin\varphi \,, \\
Z &= r \cos\theta \,,
\end{aligned} \tag{17}$$

where r,θ and φ have their usual meaning in spherical coordinate system. From (15) and (17) one finds

$$U^2 + V^2 = \ell^2 (1 + r^2/\ell^2) \,.$$

This suggests that we can parameterize U and V like

$$U = \ell\sqrt{1 + r^2/\ell^2} \cos\tau \,, \tag{18}$$
$$V = \ell\sqrt{1 + r^2/\ell^2} \sin\tau \,. \tag{19}$$

Plugging (17), (18) and (19) into (16) gives

$$\begin{aligned}
ds^2 &= -\ell^2 (1 + r^2/\ell^2) d\tau^2 - \frac{r^2 dr^2}{\ell^2 (1 + r^2/\ell^2)} + dr^2 + r^2 (d\theta^2 + \sin^2\theta d\varphi^2) \,, \\
&= -\ell^2 (1 + r^2/\ell^2) d\tau^2 + \frac{dr^2}{(1 + r^2/\ell^2)} + r^2 (d\theta^2 + \sin^2\theta d\varphi^2) \,.
\end{aligned} \tag{20}$$

By rescaling $\tau \to (t/\ell)$ we finally find a spherically symmetric Schwarzschild type metric for AdS$_4$

$$ds^2_{AdS_4} = -(1 + r^2/\ell^2)dt^2 + \frac{dr^2}{(1 + r^2/\ell^2)} + r^2 (d\theta^2 + \sin^2\theta d\varphi^2) \,. \tag{21}$$

This parametrization covers the entire spacetime. As can be seen from (17), (18) and (19) the topology of AdS$_4$ is S^1 (time) $\times$ R^3 (space) which signals that in AdS there are closed timelike lines. In order to get rid of this sickness one has to unwrap the S^1 and work, instead, in covering spacetime (CAdS$_4$) with topology R^4 with no closed timelike lines. In this parametrization of AdS, the spatial infinity is timelike and thus information can be lost to, or gained from it. Unfortunately any change in time coordinate to get rid of this problem will cost us to lose globally defined coordinates.

Now for the sake of generality we write the metric in the generic form of spherical symmetric metric, i.e.,

$$ds_4^2 = -\mathrm{e}^{A(r)}dt^2 + \mathrm{e}^{B(r)}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2 \tag{22}$$

where A, B are functions of r only. The non-vanishing components of Christoffel symbols are

$$\begin{aligned}
\Gamma^t_{tr} &= \frac{A'}{2} \,, \quad \Gamma^r_{tt} = \mathrm{e}^{A-B}\frac{A'}{2} \,, \quad \Gamma^r_{rr} = \frac{B'}{2} \,, \quad \Gamma^r_{\theta\theta} = -r\,\mathrm{e}^{-B} \,, \quad \Gamma^r_{\varphi\varphi} = -r\sin^2\theta\,\mathrm{e}^{-B} \\
\Gamma^\theta_{r\theta} &= \frac{1}{r} \,, \quad \Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta \,, \quad \Gamma^\varphi_{r\varphi} = \frac{1}{r} \,, \quad \Gamma^\varphi_{\theta\varphi} = \cot\theta \,,
\end{aligned} \tag{23}$$

where prime means derivative with respect to r.

The group $SO(2,3)$ is the symmetry group of the 4-dimensional Anti-de Sitter spacetime which plays the role of the 4-dimensional Poincaré group. The time translations are the $U(1)$ subgroup of rotations in the (U,V)-plane. The 10 generators denoted by J_{ab} act on the embedding coordinates and are defined by

$$J_{ab} = X_a \frac{\partial}{\partial X^b} - X_b \frac{\partial}{\partial X^a} \,, \tag{24}$$

where $X_a = \eta_{ab} X^b$. Thus the generators J_{ab} satisfy

$$[J_{ab}, J_{cd}] = \eta_{ad} J_{bc} + \eta_{bc} J_{ad} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac}. \tag{25}$$

Then $J_{\mu\nu}$ generate ordinary rotations around any points in the four space spanned by U, X, Y and Z. The translational operators are defined by

$$P_\mu = \ell^{-1} L_{\mu V}, \tag{26}$$

for $\mu = U, X, Y$ and Z. The commutation relations between $L_{\mu\nu}$ and P_μ are

$$[L_{\mu\nu}, P_\rho] = \eta_{\mu\rho} P_\nu - \eta_{\nu\rho} P_\mu,$$
$$[P_\mu, P_\nu] = \frac{1}{\ell^2} L_{\mu\nu}. \tag{27}$$

These results hold for any spin and reduce to the commutation relations for the Poincaré group when $\ell \longrightarrow \infty$. The Casimir bilinear invariant operator C_2 can be written as

$$C_2 = \frac{1}{2} L_{ab} L^{ab} = \frac{1}{\ell} \frac{\partial}{\partial X^a} X_a X_b \frac{\partial}{\partial X^b} - \eta^{ab} \frac{\partial}{\partial X^a} \frac{\partial}{\partial X^b} + s(s+1), \tag{28}$$

where s is the spin. Note that the first two term in C_2 is the same as the invariant Laplace-Beltrami operator. Thus $\square_{AdS_4}$ is an invariant with respect to $SO(2,3)$ and a function of C_2 [8, 9]. For the background (21), C_2 takes the following form for the gravitons

$$C_2 = \left[1 - \frac{1}{\ell^2}(r^2 - t^2)\right]\left[-\square_s + \frac{1}{\ell^2}\left(t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r} + 1\right)^2\right] + 6, \tag{29}$$

where $\square_s$ is the Laplace-Beltrami operator in spherical coordinates.

IV. FLUCTUATIONS AROUND AdS$_4$ BACKGROUND

Perturbations can be expressed in the form of four factors, each of which is a function of the coordinates, t, r, θ, and φ; this separation is achieved by the use of generalized tensor spherical harmonics. The ten components of metric perturbations can be divided into two categories, called perturbations of odd (magnetic) and even (electric) parity, which are not mixed by tensorial operators that respect spherical symmetry. Under a rotation of the frame around the origin, the $\frac{1}{2}D(D+1)$ components of the perturbing metric transform like 3 scalars: (h_{00}, h_{01}, h_{11}), and 2 $(D-2)$-vectors: $(h_{02}, h_{03}, ..., h_{0D-1}; h_{12}, h_{13}, ..., h_{1D-1})$, and a $\frac{1}{2}(D-1)(D-2)$ component second rank tensor. For instance, when $D = 4$ we have the following blocks,

$$h_{\mu\nu} = \begin{bmatrix} \begin{array}{cc} \text{Scalars} & \text{Vectors} \\ \boxed{\begin{array}{cc} h_{00} & h_{01} \\ h_{10} & h_{11} \end{array}} & \boxed{\begin{array}{cc} h_{02} & h_{03} \\ h_{12} & h_{13} \end{array}} \\ \text{Vectors} & \text{Tensors} \\ \boxed{\begin{array}{cc} h_{20} & h_{21} \\ h_{30} & h_{31} \end{array}} & \boxed{\begin{array}{cc} h_{22} & h_{23} \\ h_{32} & h_{33} \end{array}} \end{array} \end{bmatrix}, \tag{30}$$

whereas for $D = 5$ we get,

$$h_{AB} = \begin{bmatrix} \begin{array}{cc} \text{Scalars} & \text{Vectors} \\ \boxed{\begin{array}{cc} h_{00} & h_{01} \\ h_{10} & h_{11} \end{array}} & \boxed{\begin{array}{ccc} h_{02} & h_{03} & h_{04} \\ h_{12} & h_{13} & h_{14} \end{array}} \\ \text{Vectors} & \text{Tensors} \\ \boxed{\begin{array}{cc} h_{20} & h_{21} \\ h_{30} & h_{31} \\ h_{40} & h_{41} \end{array}} & \boxed{\begin{array}{ccc} h_{22} & h_{23} & h_{24} \\ h_{32} & h_{33} & h_{34} \\ h_{42} & h_{43} & h_{44} \end{array}} \end{array} \end{bmatrix}. \tag{31}$$

Associated with any perturbation modes, there is the angular momentum l and its projection on the z axis m. The general perturbation is expressed by $h_{\mu\nu} = \sum_{lm}[h_{\mu\nu}^{(odd)lm} + h_{\mu\nu}^{(even)lm}]$, where $h_{\mu\nu}^{(odd)lm}$ and $h_{\mu\nu}^{(even)lm}$ behave differently

under parity change: $\mathcal{P}[h_{\mu\nu}^{lm}(t,r,\theta,\varphi)] \rightarrow \tilde{h}_{\mu\nu}^{lm}(t,r,\pi-\theta,\pi+\varphi)$. In practice, a tensor harmonic is *odd* or *axial* if $\mathcal{P}(h_{\mu\nu}) = \tilde{h}_{\mu\nu} = (-1)^{l+1}h_{\mu\nu}$ and is *even* or *polar* if $\mathcal{P}(h_{\mu\nu}) = \tilde{h}_{\mu\nu} = (-1)^l h_{\mu\nu}$. In four dimensions, as we mentioned above, we have three scalars under rotation,

$$s_{\mu\nu}(t,r,\theta,\phi) = \sum_{lm} \alpha_{\mu\nu}^{lm}(t,r) Y_l^m(\theta,\varphi), \tag{32}$$

where $Y_l^m(\theta,\varphi)$s are well known scalar spherical harmonics and are given by

$$Y_l^m(\theta,\varphi) = \left[\frac{(2l+1)(l-m)}{4\pi(l+m)!}\right]^{(1/2)} P_l^m(\cos\theta)\mathrm{e}^{im\varphi}, \tag{33}$$

and

$$\alpha_{\mu\nu}^{lm}(t,r) = e^A H_0^{lm}(t,r)\delta^t{}_\mu \delta^t{}_\nu + H_1^{lm}(t,r)(\delta^t{}_\mu\delta^r{}_\nu + \delta^r{}_\mu\delta^t{}_\nu) + e^B H_2^{lm}(t,r)\delta^r{}_\mu\delta^r{}_\nu. \tag{34}$$

For vector spherical harmonics [13] we have two distinct types with opposite parities, i.e,

$$Y_a^{ml}(\theta,\varphi) = \begin{cases} [l(l+1)]^{(-1/2)}\frac{\partial Y_l^m}{\partial x^a}; & \text{with } \mathcal{P} = (-1)^l \\ [l(l+1)]^{(-1/2)}\epsilon_a{}^b\frac{\partial Y_l^m}{\partial x^b}, & \text{with } \mathcal{P} = (-1)^{l+1} \end{cases} \tag{35}$$

where ϵ_{ab} is totally antisymmetric tensor which is covariantly constant on S^2, i.e, $\epsilon_{bc\,;a} = 0$ and is defined by,

$$\begin{aligned} \epsilon_{\theta\theta} &= \epsilon_{\varphi\varphi} = 0 \\ \epsilon_{\theta\varphi} &= -\epsilon_{\varphi\theta} = \sin\theta. \end{aligned} \tag{36}$$

Here the lower case Latin letters a, b, and c run over the values θ and φ.

The vector part of the metric then is given by

$$v_{\mu\nu}(t,r,\theta,\varphi) = \sum_{lm} \left[\alpha_{\mu\nu}^{(lm)\,a}(t,r)\,Y_a^{lm(odd)}(\theta,\varphi) + \beta_{\mu\nu}^{(lm)\,a}(t,r)\,Y_a^{lm(even)}(\theta,\varphi)\right], \tag{37}$$

where

$$\begin{aligned} \alpha_{\mu\nu}^{(lm)\,a}(t,r) &= \sqrt{l(l+1)}\left[h_0^{lm(odd)}(t,r)\delta^t{}_\mu + h_1^{lm(odd)}(t,r)\delta^r{}_\mu\right]\delta^a{}_\nu \\ &+ \sqrt{l(l+1)}\,\delta^a{}_\mu\left[h_0^{lm(odd)}(t,r)\delta^t{}_\nu + h_1^{lm(odd)}(t,r)\delta^r{}_\nu\right], \end{aligned} \tag{38}$$

and

$$\begin{aligned} \beta_{\mu\nu}^{(lm)\,a}(t,r) &= \sqrt{l(l+1)}\left[h_0^{lm(even)}(t,r)\delta^t{}_\mu + h_1^{lm(even)}(t,r)\delta^r{}_\mu\right]\delta^a{}_\nu \\ &+ \sqrt{l(l+1)}\,\delta^a{}_\mu\left[h_0^{lm(even)}(t,r)\delta^t{}_\nu + h_1^{lm(even)}(t,r)\delta^r{}_\nu\right], \end{aligned} \tag{39}$$

with a runs over θ and φ.

For a rank-2 symmetric tensor, there are three fundamental types of tensor of angular momentum l[10],

$$Y_{ab}^{lm} = \begin{cases} \psi_{ab}^{lm} &= (1/\sqrt{l(l+1)[l(l+1)-1]})\,Y^m{}_{l\,|ab}; & \text{with } \mathcal{P} = (-1)^l \\ \phi_{ab}^{lm} &= \frac{1}{\sqrt{2}}\gamma_{ab}Y_l^m; & \text{with } \mathcal{P} = (-1)^l \\ \chi_{ab}^{lm} &= \sqrt{\frac{(l-2)!}{2(l+2)!}}\left[\epsilon_a{}^c\,Y^m{}_{l\,|cb} + \epsilon_b{}^c\,Y^m{}_{l\,|ca}\right]; & \text{with } \mathcal{P} = (-1)^{l+1} \text{ and zero trace}, \end{cases} \tag{40}$$

where $\gamma_{ab} = g_{ab}/r^2$ is the metric tensor and the | denotes covariant derivative on S^2. The tensor part of the metric then takes the following form

$$t_{\mu\nu}(t,r,\theta,\varphi) = \sum_{lm} \left[\alpha_{\mu\nu}^{(lm)\,ab}(t,r)\chi_{ab}^{lm(odd)}(\theta,\varphi) + \beta_{\mu\nu}^{(lm)\,ab}(t,r)\phi_{ab}^{lm(even)}(\theta,\varphi) + \beta'^{(lm)\,ab}_{\mu\nu}(t,r)\psi_{ab}^{lm(even)}(\theta,\varphi)\right], \tag{41}$$

where

$$\alpha_{\mu\nu}^{(lm)\,ab}(t,r) = \sqrt{\frac{(l+2)!}{2(l-2)!}}h_2^{lm}(t,r)\,\delta^a{}_\mu\delta^b{}_\nu, \tag{42}$$

and

$$\beta_{\mu\nu}^{(lm)\,ab}(t,r) \;=\; r^2\sqrt{2}K^{lm}(t,r)\,\delta_\mu{}^a\,\delta_\nu{}^b, \tag{43}$$

$$\beta_{\mu\nu}'^{(lm)\,ab}(t,r) \;=\; r^2\sqrt{l(l+1)[l(l+1)-1]}\,G^{lm}(t,r)\,\delta_\mu{}^a\,\delta_\nu{}^b. \tag{44}$$

Now we can summarize the results we have just derived. The most general form of the perturbed metric for the two distinct parities are

odd parity:

$$h_{\mu\nu}^{(odd)} =
\begin{bmatrix}
0 & 0 & -h_0(r,t)\left(\frac{\partial}{\sin\theta\,\partial\varphi}\right)Y_l^m & h_0(r,t)\sin\theta\left(\frac{\partial}{\partial\theta}\right)Y_l^m \\[4pt]
0 & 0 & -h_1(r,t)\left(\frac{\partial}{\sin\theta\,\partial\varphi}\right)Y_l^m & h_1(r,t)\sin\theta\left(\frac{\partial}{\partial\theta}\right)Y_l^m \\[4pt]
h_{t\theta} & h_{r\theta} & h_2(r,t)\left[\frac{\partial^2}{\sin\theta\,\partial\varphi\partial\theta}-\cos\theta\frac{\partial}{\sin^2\theta\,\partial\varphi}\right]Y_l^m & h_{\varphi\theta} \\[4pt]
h_{t\varphi} & h_{r\varphi} & \frac{1}{2}h_2(r,t)\left[\frac{\partial^2}{\sin\theta\,\partial\varphi^2}+\cos\theta\frac{\partial}{\partial\theta}-\sin\theta\frac{\partial^2}{\partial\theta^2}\right]Y_l^m & -h_2(r,t)\left[\sin\theta\frac{\partial^2}{\partial\theta\partial\varphi}-\cos\theta\frac{\partial}{\partial\varphi}\right]Y_l^m
\end{bmatrix} ; \tag{45}$$

even parity:

$$h_{\mu\nu}^{(even)} =
\begin{bmatrix}
e^A H_0(t,r)Y_l^m & H_1(t,r)Y_l^m & h_0(r,t)\left(\frac{\partial}{\partial\theta}\right)Y_l^m & h_0(r,t)\left(\frac{\partial}{\partial\varphi}\right)Y_l^m \\[4pt]
H_1(t,r)Y_l^m & e^B H_2(t,r)Y_l^m & h_1(r,t)\left(\frac{\partial}{\partial\theta}\right)Y_l^m & h_1(r,t)\left(\frac{\partial}{\partial\varphi}\right)Y_l^m \\[4pt]
h_{t\theta} & h_{r\theta} & r^2\left[K(t,r)+G(t,r)\frac{\partial^2}{\partial\theta^2}\right]Y_l^m & h_{\varphi\theta} \\[4pt]
h_{t\varphi} & h_{r\varphi} & r^2 G(t,r)\left[\frac{\partial^2}{\partial\theta\partial\varphi}-\cos\theta\frac{\partial}{\sin\theta\,\partial\varphi}\right]Y_l^m & r^2\left[K(t,r)\sin^2\theta+G(t,r)\left(\frac{\partial^2}{\partial\varphi^2}+\sin\theta\cos\theta\frac{\partial}{\partial\theta}\right)\right]Y_l^m
\end{bmatrix} . \tag{46}$$

A. Magnetic (odd) modes

We can now simplify the perturbations by the gauge transformation due to the coordinates transformation $\hat{x}^\mu = x^\mu + \xi^\mu$ with $\xi^\mu \ll x^\mu$:

$$\begin{aligned}
\hat{h}_{\mu\nu}(x^\rho) &= h_{\mu\nu}(x^\rho) - \xi_{\mu;\nu} - \xi_{\nu:\mu} \\
&= h_{\mu\nu}(x^\rho) - g_{\sigma\nu}\xi^\sigma{}_{,\mu} - g_{\mu\sigma}\xi^\sigma{}_{,\nu} - \xi^\sigma g_{\mu\nu,\sigma}.
\end{aligned} \tag{47}$$

We choose a gauge transformation that eliminates those terms which contain the derivatives of the highest order with respect to angles [19]:

$$\xi^\mu_{(odd)} = A(t,r)\delta^\mu{}_a \epsilon^{ab}\frac{\partial}{\partial x^b}Y_l^m \quad \text{(with } a,b = \theta,\varphi). \tag{48}$$

This specialization is accomplished by demanding radial function $A(t,r)$ to annul the radial factor $h_2(t,r)$, i.e, by choosing $A = \frac{1}{2}(h_2^{(odd)}/r^2)$. Also since the final result is independent of particular value of m, we choose $m = 0$. Finally we take the time dependence of the perturbations as $\exp(-i\omega t)$, since the background is independent of time. Note that purely positive imaginary frequencies that make perturbations grow exponentially with time. Thus the non-trivial *odd* modes are taken to be

$$h_{t\varphi}^{(odd)} = h_0(r)\,f(\theta)\,e^{-i\omega t}, \quad h_{r\varphi}^{(odd)} = h_1(r)\,f(\theta)\,e^{-i\omega t} \tag{49}$$

where, $f(\theta) = \sin\theta\,(d/d\theta)P_l(\theta)$. Thus the *odd* part of the metric is

$$h_{\mu\nu}^{(odd)} =
\begin{bmatrix}
0 & 0 & 0 & h_0 \\
0 & 0 & 0 & h_1 \\
0 & 0 & 0 & 0 \\
h_0 & h_1 & 0 & 0
\end{bmatrix} e^{-i\omega t}\,f(\theta). \tag{50}$$

There are ten equations of motion but only three of them are non-trivial. The non-trivial equations of motion according to (13) are

$$\delta R_{r\varphi} + \mathrm{L}_4 h_{r\varphi} = 0 \;\rightarrow\; \left\{\left[\frac{1}{r}(A'-B')e^{-B} + \omega^2 e^{-A} + \frac{2}{r^2}e^{-B} - \frac{6}{\ell^2} - \frac{l(l+1)}{r^2}\right]h_1 \right.$$

$$+\mathrm{i}\,\omega\mathrm{e}^{-A}\left(\frac{2}{r}-\frac{d}{dr}\right)h_0\Bigg\}\sin\theta\,\frac{dP_l(\theta)}{d\theta}\,\mathrm{e}^{-\mathrm{i}\omega t}=0\,,\tag{51}$$

$$\delta R_{\theta\varphi}+\mathrm{L}_4 h_{\theta\varphi}=0\;\rightarrow\;\Bigg\{\mathrm{i}\omega\,h_0\,\mathrm{e}^{-A}+\mathrm{e}^{-B}\left[\frac{A'-B'}{2}+\frac{d}{dr}\right]h_1\Bigg\}\times\left(\cos\theta\,\frac{d}{d\theta}-\sin\theta\,\frac{d^2}{d\theta^2}\right)P_l(\theta)\,\mathrm{e}^{-\mathrm{i}\omega t}\,,\tag{52}$$

$$\delta R_{t\varphi}+\mathrm{L}_4 h_{t\varphi}=0\;\rightarrow\;\Bigg\{\mathrm{e}^{-B}\left[\frac{d^2}{dr^2}-\frac{A'+B'}{2}\frac{d}{dr}-\frac{l(l+1)}{r^2}\,\mathrm{e}^B+\frac{2}{r}A'-\frac{3}{\ell^2}\,\mathrm{e}^B\right]h_0$$

$$+\mathrm{i}\,\omega\,\mathrm{e}^{-B}\left[\frac{d}{dr}+\frac{2}{r}-\frac{A'+B'}{2}\right]h_1\Bigg\}\sin\theta\,\frac{dP_l(\theta)}{d\theta}\,\mathrm{e}^{-\mathrm{i}\omega t}=0\,.\tag{53}$$

I. Special case, $l=0$:

All the angular factors in Eqs. (51), (52), and (53) are identical to zero for $l=0$, and since $f(\theta)=0$ it is clear that $[h_{\mu\nu}^{l=0,m=0}]^{(odd)}=0$.

II. Special case, $l=1$:

For $l=1$, however, $\delta R_{\theta\varphi}$ is trivial and $\delta R_{r\varphi}$ yields the following relation between h_0 and h_1,

$$h_1=\frac{\mathrm{i}}{\omega}r^2\frac{d}{dr}\left(\frac{h_0}{r^2}\right),\tag{54}$$

where the time dependence of the perturbations is retained as $\exp(-\mathrm{i}\omega t)$. Now one can show that these two related modes can be gauged away through gauge transformation (48), by choosing A to be $(\mathrm{i}/\omega)h_0$:

$$\xi_\varphi=\left(\frac{\mathrm{i}}{\omega}\right)h_0\exp(-\mathrm{i}\omega t)\sin\theta\frac{d}{d\theta}P_1(\cos\theta)\,.\tag{55}$$

This gauge transformation leaves other perturbation components unaltered. Thus $[h_{\mu\nu}^{l=1,m}]^{(odd)}=0$.

III. Case $l\geq 2$:

From Eq. (51) and (52), one can eliminate function h_0 in terms of h_1. The result is

$$\mathrm{e}^{-B}\left[\frac{d^2}{dr^2}+\frac{A''-B''}{2}+\frac{3(A'-B')}{2}\frac{d}{dr}+\frac{(A'-B')^2}{2}-\frac{2}{r^2}\frac{d}{dr}+\frac{2}{r^2}\right]h_1+\left[-\frac{3}{\ell^2}-\frac{l(l+1)}{r^2}+\mathrm{e}^{-A}\omega^2\right]h_1=0\tag{56}$$

Define

$$\mathcal{M}\;=\;\mathrm{e}^{\frac{1}{2}(A-B)}\,\frac{h_1}{r}\,,$$

$$dr^*\;=\;\mathrm{e}^{\frac{1}{2}(B-A)}\,dr\,,\tag{57}$$

so that

$$\frac{d^2\mathcal{M}}{dr^{*2}}\;=\;\frac{\mathrm{e}^{\frac{3}{2}(A-B)}}{r}\left[h_1''+\frac{3(A'-B')}{2}h_1'+\frac{A''-B''}{2}h_1+\frac{2}{r^2}h_1+\frac{(A'-B')^2}{2}h_1-\frac{2}{r}h_1'\right]$$

$$-\mathrm{e}^{\frac{3}{2}(A-B)}\frac{3(A'-B')}{2r}h_1\,.\tag{58}$$

Using (57) and (58), we rewrite Eq. (56) as follows

$$\frac{d^2\mathcal{M}}{dr^{*2}}+(\omega^2-V_{\mathrm{eff}}^{odd})\,\mathcal{M}=0\,,\tag{59}$$

where

$$V_{\mathrm{eff}}^{odd}=\frac{6}{\ell^2}\,\mathrm{e}^A+\frac{l(l+1)}{r^2}\,\mathrm{e}^A-\frac{3(A'-B')}{2r}\mathrm{e}^{A-B}\,.\tag{60}$$

For the AdS_4, one has

$$\mathrm{e}^A=1+\frac{r^2}{\ell^2}=\mathrm{e}^{-B}\,,\tag{61}$$

with

$$r = \ell \tan\left(\frac{r^*}{\ell}\right),\tag{62}$$

which means $r^* \in [0, \pi/2]$. So the effective potential can be written as

$$
\begin{aligned}
V_{\text{eff}}^{odd} &= \left(1 + \frac{r^2}{\ell^2}\right)\left[\frac{6}{\ell^2} + \frac{l(l+1)}{r^2} - \frac{6}{\ell^2}\right] \\
&= \left(1 + \frac{r^2}{\ell^2}\right)\frac{l(l+1)}{r^2}.
\end{aligned}\tag{63}
$$

Thus the effective potential V_{eff}^{odd} is real and positive everywhere

B. Even (electric) modes

For the electric parity, in order to fix the coordinates to the first order, we demand that the even parity functions h_0, h_1 and G vanish. This can be achieved by the following gauge transformations:

$$\xi_{even}^t = e^{-A}\left(\frac{1}{2}r^2\frac{\partial G(t,r)}{\partial t} - h_0\right)Y_l^M(\theta, \varphi),\tag{64}$$

$$\xi_{even}^r = e^{A}\left(-\frac{1}{2}r^2\frac{\partial G(t,r)}{\partial t} + h_1\right)Y_l^M(\theta, \varphi),\tag{65}$$

$$\xi_{even}^\theta = \frac{1}{2}G(t,r)\frac{\partial Y_l^M(\theta, \varphi)}{\partial \theta},\tag{66}$$

$$\xi_{even}^\varphi = \frac{1}{2}G(t,r)\frac{1}{\sin^2\theta}\frac{\partial Y_l^M(\theta, \varphi)}{\partial \varphi}.\tag{67}$$

Therefore the perturbed metric for the even parity waves in the canonical form (and by specializing to $m = 0$) is

$$h_{\mu\nu}^{even} = \begin{bmatrix} H_0(r)\,e^A & H_1(r) & 0 & 0 \\ H_1(r) & H_2(r)\,e^B & 0 & 0 \\ 0 & 0 & K(r)r^2 & 0 \\ 0 & 0 & 0 & K(r)\,r^2\sin^2\theta \end{bmatrix} e^{-i\omega t}\,P_L(\theta).\tag{68}$$

There are 7 non-trivial equations of motion for four unknowns (H_0, H_1, H_2, and K) which correspond to four diagonal components of Ricci tensor and three off-diagonal components of Ricci tensor ($(t,r), (t,\theta), (r,\theta)$). Explicitly, they are

$$
\begin{aligned}
\delta R_{tr} + Ł_4 h_{tr} &= \left\{ i\omega\left(K' - \frac{1}{2}KA' + \frac{K}{r} - \frac{1}{r}H_2\right)\right. \\
&\left. + e^{-B}\left[\frac{L(L+1)}{2r^2}e^B + \frac{3}{\ell^2}e^B - \frac{A'^2}{4} + \frac{A'B'}{4} - \frac{A'}{r} - \frac{A''}{2}\right]H_1\right\}P_L(\theta)\,e^{-i\omega t} = 0
\end{aligned}\tag{69}
$$

$$\delta R_{t\theta} + Ł_4 h_{t\theta} = \left\{i\omega\left(K + H_2\right) + \frac{A' - B'}{2}H_1\,e^{-B} + H_1'\,e^{-B}\right\}\frac{dP_L(\theta)}{d\theta}\,e^{-i\omega t} = 0\tag{70}$$

$$
\begin{aligned}
\delta R_{r\theta} + Ł_4 h_{r\theta} &= \left\{i\omega H_1\,e^{-A} + H_0' - K' + \left(\frac{A'}{2} - \frac{1}{r}\right)H_0\right. \\
&\left. + \left(\frac{A'}{2} + \frac{1}{r}\right)H_2\right\}\frac{dP_L(\theta)}{d\theta}\,e^{-i\omega t} = 0
\end{aligned}\tag{71}
$$

$$\delta R_{tt} + Ł_4 h_{tt} = \left\{\frac{1}{2}\omega^2(H_2 + 2K) + i\omega\,e^{-B}\left(\frac{B'}{2} - \frac{2}{r} - \frac{d}{dr}\right)H_1 - \frac{1}{2}e^{A-B}H_0''\right.$$

$$+ e^{A-B} \left(\frac{B'}{4} - \frac{A'}{2} - \frac{1}{r} \right) H_0' - \frac{1}{4} e^{A-B} A' H_2' + \frac{1}{2} e^{A-B} A' K'$$

$$+ e^{A-B} \left[\frac{L(L+1)}{2r^2} e^B + e^B \frac{3}{\ell^2} - \frac{A'}{r} - \frac{A'^2}{4} + \frac{A'B'}{4} - \frac{A''}{2} \right] H_0$$

$$+ e^{A-B} \left(\frac{A'B'}{4} - \frac{A''}{2} - \frac{A'^2}{4} - \frac{A'}{r} \right) H_2 \Bigg\} P_L(\theta) \, e^{-i\omega t} = 0 \,, \tag{72}$$

$$\delta R_{rr} + \mathrm{L}_4 h_{rr} = \left\{ i\omega \, e^{-A} \left(H_1' - \frac{B'}{2} H_1 \right) - \frac{1}{2} e^{B-A} \omega^2 H_2 \right.$$

$$+ \frac{H_0''}{2} - K'' + \left(\frac{A'}{2} - \frac{B'}{4} \right) H_0' + \left(\frac{A'}{4} + \frac{1}{r} \right) H_2'$$

$$+ \left(\frac{B'}{2} - \frac{2}{r} \right) K' + e^B \left[\frac{L(L+1)}{2r^2} + \frac{3}{\ell^2} \right] H_2 \Bigg\} P_L(\theta) \, e^{-i\omega t} = 0 \,, \tag{73}$$

$$\delta R_{\theta\theta} + \mathrm{L}_4 h_{\theta\theta} = \delta R_{\varphi\varphi} + \mathrm{L}_4 h_{\varphi\varphi} =$$

$$= \left\{ -\frac{\omega^2 r^2}{2} e^{-A} K + i\omega e^{-A-B} r H_1 - \frac{1}{2} e^{-B} r^2 K'' + \frac{1}{2} r e^{-B} H_0' + \frac{1}{2} r e^{-B} H_2' \right.$$

$$+ e^{-B} \left(\frac{r^2 B'}{4} - \frac{r^2 A'}{4} - 2r \right) K' + e^{-B} \left(\frac{rA'}{2} - \frac{rB'}{2} + 1 \right) H_2$$

$$+ e^{-B} \left[\frac{L(L+1)}{2} e^B + \frac{3}{\ell^2} e^B - 1 + \frac{r(B' - A')}{2} \right] K \Bigg\} P_L(\theta) \, e^{-i\omega t}$$

$$+ \frac{1}{2} (H_0 - H_2) \frac{d^2 P_L(\theta)}{d\theta^2} e^{-i\omega t} = 0 \,. \tag{74}$$

I. Special case, $l = 0$:

Making a gauge transformation $\xi_{even}^t = E_0(t,r) Y_0^0$ and $\xi_{even}^r = E_1(t,r) Y_0^0$ we can choose $E_0(r,t)$ and $E_1(r,t)$ such that $H_1(r,t) = 0 = K(r,t)$, i.e.,

$$\xi^t = \int dr' e^{-A(r)} \left(\frac{1}{2} r e^{-A(r')} \frac{\partial K(t,r')}{\partial t} - H_1(t,r') \right) + C(t) \,, \tag{75}$$

$$\xi^r = \frac{1}{2} r K(t,r) \,. \tag{76}$$

Since Y_0^0 is a number, the only trivial magnetic equations for $l = 0$ are δR_{tr}, δR_{tt}, δR_{rr}. Equation $\delta R_{\theta\theta} = 0$ is satisfied identically by a solution of other three equations. Equation $\delta R_{tr} = 0$ assures us that $H_2(t,r) = H_2(r)$ and $\delta R_{tt} + \delta R_{rr} = 0$ gives $\partial H_0(t,r)/\partial r = dH_2(r)/dr$, i.e.,

$$H_0(r,t) = H_2(r) + c(t) \,, \tag{77}$$

where $c(t)$ is some arbitrary function of time and can be removed by a suitable gauge transformation of the form $\xi^t = \frac{1}{2} Y_0^0 \int c(t') dt'$. Thus $H_0(r) = H_2(r)$.

Note that for the AdS_4 background we have

$$A'' + A'^2 + \frac{2}{r} A' = \frac{6}{\ell^2} e^{-A} \,.$$

Equation $\delta R_{\theta\theta} = 0$ now gives

$$H_0(r) = H_2(r) = \frac{c}{r(1 + r^2/\ell^2)} \,, \tag{78}$$

where c is a constant which has to be determined according to the boundary conditions. Though Eq. (78) well-behaves at infinity it blows up at the origin, so the suitable choice for the c is to set it to zero which means,

$$H_0 = H_2 = 0 \longrightarrow [h_{\mu\nu}^{l=0,m=0}]^{(even)} = 0 \,. \tag{79}$$

This proves that the monopole perturbation on AdS_4 is zero as one may expect due to the absence of massive perturbation. So the conclusion is, for AdS_4 background, $h_{\mu\nu}^{00} = 0$.

II. Special case: $l = 1$:

For $l = 1$ one can readily see from (46) that there are only six independent radial functions in the general form, i.e., the triviality of the angular part in $h_{\theta\varphi} = h_{\varphi\theta}$ leaves $G^{lm}(r,t)$ totally undetermined:

$$\frac{\partial}{\partial\varphi}\left(\frac{\partial}{\partial\theta} - \cot\theta\right)Y_1^m \equiv 0 \Longrightarrow h_{\theta\varphi} = h_{\varphi\theta} = 0 \ .$$

This gives us an additional degree of freedom while making the gauge transformation which can be utilize to impose the condition $G = K$ to annul $h_{\theta\theta}$ and $h_{\varphi\varphi}$ components:

$$h_{aa} = g_{aa}[K - G]Y_1^m \Longrightarrow K_1 = K - G = 0, \qquad \text{for } G = K \ .$$

Thus in the canonical gauge there are only three radial functions, namely, H_0, H_1 and H_2 that need to be determined. From δR_{tr} one can infer that

$$H_1 = i\omega r H_2 \ ,$$

for $l = 1$. Combining this with eq. (70) gives H_2,

$$H_2(r) = \frac{\kappa}{r^2\sqrt{1 + r^2/\ell^2}} \ , \tag{80}$$

where κ is a constant need to be determined according to the boundary conditions. It can be clearly seen that as $r \to \infty$, $H_2(r) \propto (1/r^3)$ approaches zero ($h_{11} \propto 1/r^4 \to 0$). At the origin H_2 (h_{11}) blows up and thus κ is set to zero. So for $l = 1$,

$$H_2 = 0 = H_1 \ . \tag{81}$$

Now it is easy to find H_0 from the remaining equations:

$$H_0(r) = \frac{cr}{\sqrt{1 + r^2/\ell^2}} \ . \tag{82}$$

As one can observe H_0 approaches the constant c at infinity (whereas $h_{00} \propto r \to \infty$) and falls to zero at the origin. The best choice for c is then to set it to zero and hence $H_2 = 0$. So in short,

$$[h_{\mu\nu}^{l=1,m}]^{(even)} = 0 \longrightarrow [h_{\mu\nu}^{1m}]_{AdS_4} = 0 \ , \tag{83}$$

which means there is no electric dipole as well as monopole perturbation around AdS_4 background and all radiative modes have $l \geq 2$.

III. Case $l \geq 2$:

Specializing to the AdS_4 background, the case $l > 1$ corresponds to seven radial functions, which in the canonical gauge, reduce to only four, i.e., H_0, H_1, H_2, and K. In this case $\delta R_{\mu\nu} + Ł_4 h_{\mu\nu} = 0$ implies all the bracket factors in front of the angular coefficients vanish. The field equation $\delta R_{\theta\theta} + Ł_4 h_{\theta\theta} = 0$ (or $\delta R_{\varphi\varphi} + Ł_4 h_{\varphi\varphi} = 0$) yields the relation

$$H_0 = H_2 \equiv H \ . \tag{84}$$

Eliminating H_0 from the other six equations gives the following system of radial equations:

$$(t,r) \ : \ i\omega\left(K' - \frac{1}{2}KA' + \frac{1}{r}K - \frac{1}{r}H\right) + \frac{L(L+1)}{2r^2}H_1 = 0 \ , \tag{85}$$

$$(t,\theta) \ : \ i\omega(K + H) + A'H_1 e^A + H_1' e^A = 0 \ , \tag{86}$$

$$(r,\theta) \ : \ i\omega H_1 e^{-A} + H' - K' + A'H = 0 \ , \tag{87}$$

$$(t,t) \; : \; \frac{1}{2}\,\omega^2(H+2K) - i\omega\,e^A\left(\frac{A'}{2} + \frac{2}{r} + \frac{d}{dr}\right)H_1 - \frac{1}{2}\,H''$$

$$+ e^{2A}\left(\frac{A'}{2} + \frac{1}{r}\right)H' + \frac{1}{2}\,e^{2A}\,A'K' + e^A\left[\frac{L(L+1)}{2r^2} - \frac{3}{\ell^2}\right]H = 0\,, \tag{88}$$

$$(r,r) \; : \; i\omega\,e^{-A}\left(H_1' + \frac{A'}{2}H_1\right) - e^{2A}\,\frac{\omega^2}{2}\,H$$

$$+ \frac{H''}{2} - K'' + A'\,H' + \frac{1}{r}\,H' - \left(\frac{A'}{2} + \frac{2}{r}\right)K' + e^{-A}\left[\frac{L(L+1)}{2r^2} + \frac{3}{\ell^2}\right] = 0\,, \tag{89}$$

$$(\theta,\theta) \; : \; -\frac{\omega^2 r^2}{2}e^{-A}K + i\omega\,r\,H_1 - \frac{1}{2}\,e^A r^2\,K'' + r e^A\,H'$$

$$- e^A\left(\frac{r^2 A'}{2} + 2r\right)K' + e^A\,(rA'+1)\,H + e^A\left[\frac{L(L+1)}{2}\,e^{-A} - \frac{3}{\ell^2}\,e^{-A} - 1 - rA'\right]K = 0\,. \tag{90}$$

Notice that from Eqs.(85), (86) and (87) one can derive any of the second order equations (88), (89) and (90) provided the following algebraic equation

$$\mathcal{G}(r) \; = \; \left[-2\,e^A + L(L+1) + rA'\right]H + \frac{i}{2\omega}\left[-4rw^2 + e^A\,L(L+1)A'\right]H_1$$

$$+ \left[-\frac{6r^2}{\ell^2} + 2\,e^A - L(L+1) + 2r^2\omega^2\,e^{-A} + e^A\,rA' + e^A\,\frac{r^2 A'^{\,2}}{2}\right]K = 0 \tag{91}$$

Thus the original system of radial functions has been reduced to a system of two first-order equations for two unknown functions. We can proceed to write this as a single second-order equation following [15] by introducing new function $\mathcal{E}(r)$ through the following relations

$$K = \frac{l(l+1)}{r}\mathcal{E} + \frac{d\mathcal{E}}{dr^*}\,, \tag{92}$$

$$H_1 = -i\omega\frac{\ell^2}{\ell^2 + r^2}\mathcal{E} - i\omega\frac{\ell^2 r}{\ell^2 + r^2}\frac{d\mathcal{E}}{dr^*}\,. \tag{93}$$

So now the Einstein equations for the perturbed metric (46) leads to the following equation

$$\frac{d^2\mathcal{E}}{dr^{*2}} + [\omega^2 - V_{\text{eff}}^{even}(r)]\mathcal{E} = 0\,, \tag{94}$$

where,

$$dr^* = e^{-A}\,dr\,, \tag{95}$$

with

$$V_{\text{eff}}^{even} = \left(1 + \frac{r^2}{\ell^2}\right)\frac{l(l+1)}{r^2}\,. \tag{96}$$

Notice that the effective potential in both the odd and even cases are identical unlike the case of pure Schwarzschild [15] or Schwarzschild-AdS$_4$ [16] cases. This should not surprise us as in the absence of any gravitational waves scatterer (namely massive source) there should be no difference between odd and even modes. Note that the only invariant of the group is the Laplace-Beltrami operator which is the wave equation for the free field. So, in this case we could not have two wave equations due the symmetry of the AdS$_4$ group.

V. STABILITY ANALYSIS

In order to look at the stability of the modes we should solve Eqs. (59) and (94). For that let's first write these Schrödinger-type equations in the following form

$$e^{2A}\frac{d^2\Psi(r)}{dr^2} + A'e^{2A}\frac{d\Psi(r)}{dr} + e^A\left[\frac{\omega^2 r^2 e^{-A} - l(l+1)}{r^2}\right]\Psi(r) = 0\,. \tag{97}$$

where $\Psi(r)$ is a generic form for $\mathcal{M}(r)$ or $\mathcal{E}(r)$. There are two conditions for stability which have to be satisfied at once: (1) The frequency of the modes has to be non-positive imaginary and (2) the modes (Ψ here) should satisfy the completeness condition, i.e., the norm of the solution to Eq. (97) should be finite. Now to solve Eq. (97) we have to deal with the boundary conditions of the AdS, particularly the one at infinity. As we have mentioned earlier there are problems associated with the non-hyperbolicity of the Anti-de Sitter spacetime. There have been many attempts to address the ambiguity of the AdS boundary condition at infinity [1, 2, 3, 4]. We shall adopt the boundary condition introduced in [1] in which the mode function dies off at infinity. We also demand that Ψ to be well-defined everywhere including at the origin.

Defining a dimensionless quantity a such that

$$\ell^2 a = r^2 \mathrm{e}^{-A} = \frac{r^2}{1 + r^2/\ell^2}\,, \tag{98}$$

will change (97) to the following dimensionless second order equation

$$4a(1-a)\frac{d^2\Psi(a)}{da^2} + 2(1-2a)\frac{d\Psi(a)}{da} + \frac{\omega^2\ell^2 a - l(l+1)}{a}\Psi(a) = 0\,. \tag{99}$$

Note that the range of a is finite, i.e. $a \in [0,1]$ (a is like the normalized r^*). We can hope to obtain a hypergeometric differential equation by rescaling $\psi(a)$

$$\Psi(a) = \sqrt{a-1}\, a^{(l+1)/2}\Phi(a)\,, \tag{100}$$

with

$$\frac{d\Psi}{da} = \sqrt{a-1}\, a^{(l+1)/2}\left\{\frac{d\Phi}{da} + \frac{1}{2}\left[\frac{x+(l+1)(x-1)}{x(x-1)}\right]\Phi\right\}\,, \tag{101}$$

$$\frac{d^2\Psi}{da^2} = \sqrt{a-1}\, a^{(l+1)/2}\left\{\frac{d^2\Phi}{da^2} + \left[\frac{x+(l+1)(x-1)}{x(x-1)}\right]\frac{d\Phi}{da} + \left[\frac{(l^2-1)}{4a^2} + \frac{l+1}{2a(a-1)} - \frac{1}{4(a-1)^2}\right]\Phi\right\}\,. \tag{102}$$

Therefore if $\Phi(a)$ is well-defined everywhere in the range of $[0,1]$ then (100) guaranties that $\psi(a)$ is zero at both the origin and the infinity of the AdS_4. Substituting (100) into (99) yields

$$4a(1-a)\frac{d^2\Phi(a)}{da^2} + [6 - 4l(a-1) - 12a]\frac{d\Phi(a)}{da} + [\omega^2\ell^2 a - l^2 - 4(l+1)]\Psi(a) = 0\,, \tag{103}$$

which has a solution of the form

$$\Phi(a) = C_1\, {}_2F_1\left[1 + \frac{l}{2} - \frac{\omega\ell}{2}, 1 + \frac{l}{2} + \frac{\omega\ell}{2}; \frac{3}{2} + l; a\right] + C_2\, {}_2F_1\left[1 - \frac{l}{2} - \frac{\omega\ell}{2}, \frac{1}{2} - \frac{l}{2} + \frac{\omega\ell}{2}; \frac{1}{2} - l; a\right](-a)^{-(1+2l)/2}\,, \tag{104}$$

where C_1 and C_2 are constant numbers. In order to have a well-behavior $\Phi(a)$ we need to set C_2 to zero and the first argument of the hypergeometric function, ${}_2F_1$, to a negative (real) integer number such that $\Phi(a)$ becomes a polynomial of order N at $a \to 1$ ($r \to \infty$), i.e., $\Phi(a) \sim a^N \sim a^{-(1+l/2-w\ell/2)}$. This requires

$$\omega\ell = l + 2(N+1)\,, \quad \text{for } l \geq 2\,, \tag{105}$$

where N is a non-negative integer number. Thus there is a constraint on frequency of the gravitational modes in AdS_4. Frequency is quantized and it is always positive-definite (and real) which means there could be no unstable modes and hence the AdS_4 spacetime is stable against small gravitational fluctuations as was. Equation (105) also shows that there is gap in frequency (energy), i.e., it could not ever be zero. Also note that ${}_2F_1$ is complete by construction which means that all the modes are normalizable. This complete our proof of stability of AdS_4 against gravitational perturbation. Note that according to [2], for the massless minimally coupled scalar perturbation, the frequency of the modes quantized as $l + 2N + 3$.

VI. DISCUSSION

In this paper we analyzed the perturbations on the Anti-de Sitter spacetime in details. We have shown that a perturbed AdS does not admit any non-radiative modes while the final form of the governing equations for radiative

odd and even modes are identical. In proof of stability we explicitly proved the necessary and sufficient conditions of stability: non-existence of (positive) imaginary frequency and completeness of the modes. We showed that the frequency of the gravitational modes must be quantized.

Here we would like to emphasize on two implicit features in the discussion of the tensor perturbation of the AdS. First, note that Eq. (97) is a tensor equation though it may look like the equation of motion for a scalar, Ψ. In our notation Ψ (or Φ) corresponds to covariant components of the fluctuations. There is no rule to exclude other cases and it seems that is more matter of taste rather than deep physical notion. Second, we selected a boundary condition that Ref. [1] has adopted for a scalar perturbation at spatial infinity. This choice of the boundary conditions at spatial infinity by no means is gauge-invariant. What one can do at best is to fix a gauge such that all the modes die off at spatial infinity. Although one may argue that supersymmetric boundary conditions will take care of this problem but as far as the authors are aware there are some ambiguities associated with the gauge-invariant description of the boundary conditions at spatial infinity due to the lack of hyperbolicity of the AdS which results in propagation of information from and into the spatial boundary in a finite coordinate time [17]. One way of by passing the absence of any gauge-invariant boundary conditions in the AdS spacetime is to work with gauge-invariant variable quantities rather than working in a particular gauge. This issue is under investigation and we hope to address this in our future publication [18].

At the end we would like to mention that despite the complicated nature of Regge-Wheeler decomposition one may apply it to higher dimensional AdS and the cases with brane-worlds. The latter may be achieved by considering perturbation like $\delta G_{\mu\nu} = \kappa^2 \delta T_{\mu\nu} = \lambda \delta(r - r_0)\delta g_{\mu\nu}$, where λ is a constant and r_0 is the position of the brane-world.

Acknowledgments

[1] S. J. Avis, C. J. Isham and D. Storey , Phys. Rev. D **18**, 3565 (1978).
[2] C. P. Burgess and C. A. Lutken, Phys. Lett., **153B**, 137 (1985).
[3] L. F. Abbott and S. Deser, Nucl. Phys. **B195** 76 (1982).
[4] P. Breitenlohner and D. Z. Freedman, Phys. Lett. B, **115**, 197 (1982).
[5] J. M. Maldacena, Adv. Theor. Math. Phys., **2**, 231 (1998) [arXiv: hep-th/07111200].
[6] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, Phys. Lett. B, **428**, 105 (1998) [arXiv: hep-th/9802109].
[7] E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998) [arXiv: hep-th/9802109].
[8] C. Fronsdal, Phys. Rev. **D10** 589 (1974).
[9] C. Fronsdal, Phys. Rev. **D12** 3819 (1975).
[10] T. Regge and J.A. Wheeler, Phys. Rev. **108** 1063 (1957).
[11] D.R. Brill and J.B. Hartle, Phys. Rev. **135** B271 (1964) .
[12] L.A. Edelstein and C.V. Vishveshwara, Phys. Rev. **D1** 3514 (1970).
[13] K. S. Thorne, Rev. Mod. Phys., **52**, 299 (1980).
[14] A. Higuchi, J. Math. Phys., **28**, 1553 (1987).
[15] F. Zerilli, Phys. Rev. Lett., **24**, 737 (1970).
[16] V. Cardoso and J. P. S. Lemos, Phys. Rev., **D64**, 084017 (2001) [arXiv: gr-qc/0105103].
[17] A. H. Guth, D. I. Kaiser, P. Mannheim and A. Nayeri, in preparation (2005).
[18] A. Nayeri, in preparation (2005).
[19] One of the most popular gauge in studying tensor perturbation in general relativity is the so-called transverse traceless gauge, i.e., $h^{\mu\nu}{}_{;\nu} = 0 = h^{\mu}{}_{\mu}$. On S^2, however, it is well known that no second rank symmetric tensor spherical can exist [14]. Using spherical harmonics to decompose the metric in 4 dimensions do not permit us to use transverse traceless gauge.